Warwick Mathematics Exchange

MA473

Reflection Groups

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Desync, aka The Big Ree

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Introduction

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Disclaimer: I make *absolutely no guarantee* that this document is complete nor without error. In particular, any content covered exclusively in lectures (if any) will not be recorded here. This document was written during the 2023 academic year, so any changes in the course since then may not be accurately reflected.

Notes on formatting

New terminology will be introduced in *italics* when used for the first time. Named theorems will also be introduced in *italics*. Important points will be **bold**. Common mistakes will be underlined. The latter two classifications are under my interpretation. YMMV.

Content not taught in the course will be outlined in the margins like this. Anything outlined like this is not examinable, but has been included as it may be helpful to know alternative methods to solve problems.

The table of contents above, and any inline references are all hyperlinked for your convenience.

History

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This document was written by R.J. Kit L., a maths student. I am not otherwise affiliated with the university, and cannot help you with related matters.

Please send me a PM on Discord @Desync#6290, a message in the WMX server, or an email to Warwick.Mathematics.Exchange@gmail.com for any corrections. (If this document somehow manages to persist for more than a few years, these contact details might be out of date, depending on the maintainers. Please check the most recently updated version you can find.)

If you found this guide helpful and want to support me, you can buy me a coffee!

(Direct link for if hyperlinks are not supported on your device/reader: ko-fi.com/desync.)

*Storing dates in big-endian format is clearly the superior option, as sorting dates lexicographically will also sort dates chronologically, which is a property that little and middle-endian date formats do not share. See ISO-8601 for more details. This footnote was made by the computer science gang.

1 Reflection Groups

Let V be a finite-dimensional vector space over $\mathbb{R}$. A form $\langle\,\cdot\,,\cdot\,\rangle : V \times V \to \mathbb{R}$ is

- *symmetric* if $\langle x,y\rangle = \langle y,x\rangle$ for all $x,y \in V$;
- *bilinear* if $\langle \alpha x + \beta y, z\rangle = \alpha\langle x,z\rangle + \beta\langle y,z\rangle$ and $\langle x, \alpha y + \beta z\rangle = \alpha\langle x,y\rangle + \beta\langle x,z\rangle$ for all $x,y \in V$ and $\alpha,\beta \in \mathbb{R}$;
- *positive definite* if $\langle x,x\rangle \geq 0$ for all $x \in V$, with equality if and only if $x = 0_V$.

All the forms we will consider will be symmetric and bilinear.

A space equipped with a form satisfying the three properties above is called a *Euclidean space.*

Example. $V = \mathbb{R}^2$ with $\big\langle (x_1,x_2),(y_1,y_2)\big\rangle = x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2$ is a Euclidean space. △

Example. $V = \mathbb{R}^n$ with $\langle x,y\rangle = x \cdot y = \sum_i x_iy_i$ is the standard Euclidean space, denoted by $\mathbb{E}^n$. △

Via the Gram-Schmidt process, every Euclidean vector space V has an orthonormal basis – that is, a basis $(e_i)_{i=1}^n$ of unit vectors ($\|e_i\| = 1$) that are pairwise orthogonal ($e_i \cdot e_j = 0$ for $i \neq j$). Thus, we can find an isomorphism $V \to \mathbb{E}^n$ preserving the bilinear form.

Example. In the non-standard Euclidean space above,

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$$

is such a basis. △

The *general linear group* $GL(V)$ of a vector space V is the group of linear automorphisms of V with the operation of composition.

$$\begin{aligned} GL(V) &\coloneqq \operatorname{Aut}(V) \\ &= \big\{(T : V \to V) : T \text{ is linear and bijective}\big\} \end{aligned}$$

If V has dimension $n < \infty$, then this is isomorphic to the group of $n \times n$ invertible matrices with the operation of matrix multiplication.

The *orthogonal group* $O(V)$ of a vector space V is the subgroup of the general linear group consisting of transformations that preserve the bilinear form

$$O(V) \coloneqq \big\{T \in GL(V) : \big\langle T(x) \cdot T(y)\big\rangle = \langle x,y\rangle\big\}$$

Because vector norms are defined in terms of bilinear forms, e.g. $\|x\| = \sqrt{\langle x,x\rangle}$, orthogonal transformations $T \in O(V)$ also preserve vector norms: $\|x\| = \|T(x)\|$ for any $x \in V$.

Let $x \in V$ be non-zero. We define the map $S_x : V \to V$ by

$$S_x(z) \coloneqq z - 2\frac{\langle x,z\rangle}{\langle x,x\rangle}x$$

This is always an element of $O(V)$. Note also that $S_x(x) = -x$.

An element $T \in O(V)$ is a *reflection* if the set of fixed points $V^T \coloneqq \{v \in V : T(v) = v\}$ is an $(n-1)$-dimensional subspace of V.

Example. The map S_x is a reflection, and the fixed subspace is the orthogonal complement $\{x\}^\perp$. △

The next lemma shows that all reflections are of this form.

Lemma 1.1. *Let V be a Euclidean space. Let T be a reflection, and let $x \in (V^T)^\perp$ be non-zero. Then, $T(x) = -x$ and $T = S_x$.*

Proof. Let $v \in V^T$, so $v = T(v)$. Then, $\big\langle v,T(x)\big\rangle = \big\langle T(v),T(x)\big\rangle = \langle v,x\rangle = 0$, so v and $T(x)$ are orthogonal, i.e., $T(x) \in (V^T)^\perp$.

Since $\dim\big((V^T)^\perp\big) = \dim(V) - \dim(V^T) = 1$, $T(x) = \alpha x$ for some $\alpha \in \mathbb{R}$. Then, since x is non-zero,

$$\begin{aligned}\langle x,x\rangle &= \langle T(x),T(x)\rangle \\ &= \langle \alpha x,\alpha x\rangle \\ &= \alpha^2\langle x,x\rangle\end{aligned}$$

since $x \neq 0$, $\langle x,x\rangle \neq 0$, so $\alpha^2 = 1$. If $\alpha = 1$, then $T(x) = x$ and $x \in V^T$, contradicting that $x \in (V^T)^\perp$. Hence, $\alpha = -1$, so $T(x) = -x$.

Now, suppose $z \in V$. Then, by linearity in the first argument,

$$\begin{aligned}\left\langle z - \frac{\langle x,z\rangle}{\langle x,x\rangle}x,\, x\right\rangle &= \langle z,x\rangle - \frac{\langle x,z\rangle}{\langle x,x\rangle}\langle x,x\rangle \\ &= \langle z,x\rangle - \langle z,x\rangle \\ &= 0\end{aligned}$$

so $z - \frac{\langle x,z\rangle}{\langle x,x\rangle}x \in \{x\}^\perp = V^T$, and $T\big(z - \frac{\langle x,z\rangle}{\langle x,x\rangle}x\big) = z - \frac{\langle x,z\rangle}{\langle x,x\rangle}x$. So,

$$\begin{aligned}T(z) &= T\left(z - \frac{\langle z,x\rangle}{\langle x,x\rangle}x + \frac{\langle z,x\rangle}{\langle x,x\rangle}x\right) \\ &= T\left(z - \frac{\langle z,x\rangle}{\langle x,x\rangle}x\right) + \frac{\langle z,x\rangle}{\langle x,x\rangle}T(x) \\ &= z - \frac{\langle z,x\rangle}{\langle x,x\rangle}x - \frac{\langle z,x\rangle}{\langle x,x\rangle}x \\ &= z - 2\frac{\langle z,x\rangle}{\langle x,x\rangle}x \\ &= S_x(z)\end{aligned}$$

■

From this lemma, we deduce

- Every reflection T in a Euclidean space is of the form S_x for some x determined uniquely up to scaling. Such an x is called the *root* of the reflection T.
- For all non-zero $x \in V$, the map S_x is a reflection.
- Every reflection T is involutive, i.e., satisfies $T^2 = \mathrm{id}_V$.

A *finite reflection group* is a pair (G,V) consisting of a Euclidean space V and a finite subgroup $G < O(V)$ generated by reflections, e.g. $G = \big\langle\{S_x : S_x \in G\}\big\rangle$.

Example. The trivial group generated by no reflections forms the trivial reflection group $(0,V)$. △

Example. $\big(\{\mathrm{id}_\mathbb{R}, f\},\mathbb{R}\big)$ with f defined by $x \mapsto -x$ is a reflection group, with $f = S_1$. △

Two reflection groups (G_1,V_1) and (G_2,V_2) are *equivalent* if there exists an isometry $\varphi : V_1 \to V_2$ such that $\varphi G_1\varphi^{-1} := \{\varphi T\varphi^{-1} : T \in G_1\} = G_2$, written as $(G_1,V_1) \simeq (G_2,V_2)$.

Example. The reflection group $\big(\{\mathrm{id}_{\mathbb{R}^2}, S_{(0,1)}\}, \mathbb{R}^2\big)$ generated by the reflection along the x-axis and the reflection group $\big(\{\mathrm{id}_{\mathbb{R}^2}, S_{(0,1)}\}, \mathbb{R}^2\big)$ generated by the reflection along the y-axis are equivalent, with the isometry given by the rotation

$$\varphi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

△

Example. Let $x,y \in \mathbb{R}^2$ be non-zero. Consider the group $\langle S_x, S_y \rangle$ generated by the reflections S_x and S_y. As matrices, we have

$$\begin{aligned} \det(S_x S_y) &= \det(S_x)\det(S_y) \\ &= (-1)^2 \\ &= 1 \end{aligned}$$

so $S_x S_y \in SO(2)$, i.e., is a rotation by some angle α (in fact, α is twice the angle between x and y). If $\frac{\alpha}{2\pi} \notin \mathbb{Q}$, then $S_x S_y$ has infinite order and $\langle S_x, S_y \rangle$ is not finite. Otherwise, if $\frac{\alpha}{2\pi} = \frac{n}{k}$ for coprime integers $n,k \in \mathbb{Z}$, then $S_x S_y$ is a rotation by $\frac{2\pi n}{k}$ and hence has order k. From this, we deduce that $\langle S_x, S_y \rangle$ is isomorphic to the dihedral group $\mathrm{Dih}(k)$ of symmetries on the k-gon of order $2n$ (i.e. as generated by the reflection $\sigma = S_x$ and the rotation $\tau = S_x S_y$). △

We define the group $I_2(k)$ as

$$\begin{aligned} I_2(k) &:= \Big(\big\langle S_{(1,0)}, S_{\left(\cos(\frac{\pi}{k}), \sin(\frac{\pi}{k})\right)}\big\rangle, \mathbb{R}^2\Big) \\ &= \big(\mathrm{Dih}(k), \mathbb{R}^2\big) \end{aligned}$$

with the subscript matching the dimension. Note that $|I_2(k)| = 2k$.

Example. The symmetric group $\mathrm{Sym}(n)$ acts on $\{1, \ldots, n\}$. We can extend this action to $\mathbb{R}^n$ as follows. Let $(e_i)_{i=1}^n$ be a basis of $\mathbb{R}^n$. For each permutation $\sigma \in \mathrm{Sym}(n)$, define $T_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ by $T_\sigma(e_i) = e_{\sigma(i)}$. Then, $\sigma \mapsto T_\sigma$ defines a homomorphism $\mathrm{Sym}(n) \to O(\mathbb{R}^n)$ and hence a subgroup of $O(\mathbb{R}^n)$.

The symmetric group $\mathrm{Sym}(n)$ is generated by the transpositions (i,j), $i \neq j$, and $T_{(i,j)} = S_{e_i - e_j}$ is a reflection, so $T_{(i,j)}(e_i - e_j) = e_j - e_i = -(e_i - e_j)$. Now, any vector $y \in (e_i - e_j)^\perp$ will have equal i and j coordinates, and hence $T_{(i,j)}(y) = (y)$. Thus, this defines a finite reflection group $\big(\mathrm{Sym}(n), \mathbb{R}^n\big)$.

However, note that the vector $x = (1,1\ldots,1)$ is fixed by any T_σ, so the orthogonal complement $x^\perp := \big\{(a_1, \ldots, a_n) : \sum_i a_i = 0\big\}$ is also invariant (i.e. permuting the summands doesn't change its value) and is isomorphic to $\mathbb{R}^{n-1}$.

So, we can reduce $\big(\mathrm{Sym}(n), \mathbb{R}^n\big)$ to $\big(\mathrm{Sym}(n), x^\perp\big) \simeq \big(\mathrm{Sym}(n), \mathbb{R}^{n-1}\big)$ △

We define the group A_{n-1} (unrelated to the alternating group) by:

$$\begin{aligned} A_{n-1} &:= \big(\mathrm{Sym}(n), x^\perp\big) \\ &\simeq \big(\mathrm{Sym}(n), \mathbb{R}^{n-1}\big) \end{aligned}$$

again, with the subscript matching the dimension. Note that $|A_n| = (n+1)!$ and $A_2 = I_2(3)$.

We define the group B_n by modifying A_n as:

$$B_n := \big(\langle \mathrm{Sym}(n), \underset{i=1,\ldots,n}{S_{e_i}} \rangle, \mathbb{R}^n\big)$$

and also the group D_n as:

$$B_n := \big(\langle \mathrm{Sym}(n), \underset{i \neq j}{S_{e_i + e_j}} \rangle, \mathbb{R}^n\big)$$

D_n has index 2 in B_n.

2 Root Systems

Lemma 2.1. *For any $T \in O(V)$,*

$$TS_xT^{-1} = S_{T(x)}$$

Proof. First, note

$$\begin{aligned}(TS_xT^{-1})\big(T(x)\big) &= (TS_xT^{-1}T)(x)\\ &= (TS_x)(x)\\ &= T(-x)\\ &= -T(x)\end{aligned}$$

More generally,

$$\begin{aligned}(TS_xT^{-1})(z) &= T\big(S_x(T^{-1}z)\big)\\ &= T\left(T^{-1}z - 2\frac{\langle T^{-1}z,x\rangle}{\langle x,x\rangle}x\right)\\ &= T(T^{-1}z) - T\left(2\frac{\langle T^{-1}z,x\rangle}{\langle x,x\rangle}x\right)\\ &= z - 2\frac{\langle T^{-1}z,x\rangle}{\langle x,x\rangle}T(x)\\ &= z - 2\frac{\langle z,Tx\rangle}{\langle Tx,Tx\rangle}T(x)\\ &= S_{T(x)}\end{aligned}$$

where the last line follows from $T \in O(V)$ being orthogonal, i.e. $\langle u,v\rangle = \big\langle T(u),T(v)\big\rangle$. ■

The *root system* of a finite reflection group (G,V) is the set

$$\Phi_{(G,V)} \coloneqq \big\{x \in V : S_x \in G, \|x\| = 1\big\}$$

Theorem 2.2. *The root system of a finite reflection group (G,V) satisfies the following properties:*

(i) *If $x \in \Phi_{(G,V)}$, then $\mathbb{R}x \cap \Phi_{(G,V)} = \{x,-x\}$;*

(ii) *The cardinality of $\Phi_{(G,V)}$ is twice the number of reflections in G.*

(iii) *If $T \in G$ and $x \in \Phi_{(G,V)}$, then $T(x) \in \Phi_{(G,V)}$*

Proof.

(i) If $x \in \Phi_{(G,V)}$, then $\|x\| = 1$, so $\mathbb{R}x \cap \Phi_{(G,V)} = \big\{\alpha x : \alpha \in \mathbb{R}, \|x\| = 1\big\} = \{x,-x\}$.

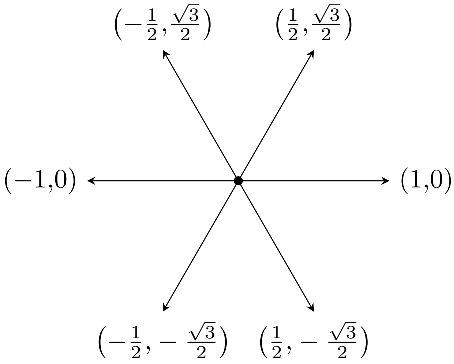
(ii) For each reflection $S_x \in G$, there are two elements $x, -x \in \Phi_{(G,V)}$.

(iii) If $x \in \Phi_{(G,V)}$, then $S_x \in G$. Then, for any $T \in G$, $TS_xT^{-1} = S_{T(x)} \in G$. Also, since $T \in G < O(V)$ is orthogonal, $\|T(x)\| = \|x\| = 1$. Hence, $T(x) \in \Phi_{(G,V)}$.

■

Example. $I_2(3)$ describes the symmetry of an equilateral triangle, so the roots are given by the unit vectors on the lines of symmetry:

$$\Phi_{I_2(3)} = \Big\{(0,1),(0,-1),\big(\tfrac{\sqrt{3}}{2},\tfrac{1}{2}\big),\big(-\tfrac{\sqrt{3}}{2},-\tfrac{1}{2}\big),\big(-\tfrac{\sqrt{3}}{2},\tfrac{1}{2}\big),\big(\tfrac{\sqrt{3}}{2},-\tfrac{1}{2}\big)\Big\}$$



△

2.1 Abstract Root Systems

More generally, any set of vectors $\Phi \subseteq V$ is a *root system* if:

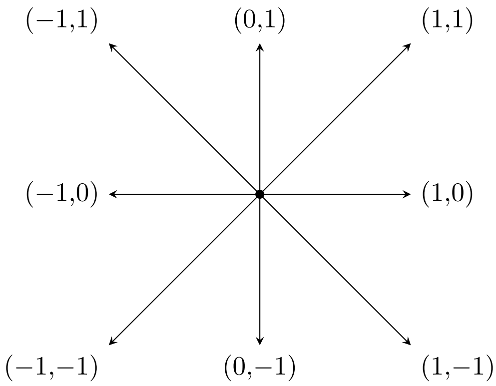
(*i*) $0_V \notin \Phi$;

(*ii*) If $x \in \Phi$, then $\mathbb{R}x \cap \Phi = \{x,-x\}$;

(*iii*) If $x,y \in \Phi$, then $S_y(x) \in \Phi$.

Note that we do not require the vectors in an abstract root system to have unit norm, unlike the root system associated to a finite reflection group.

Example. $\Phi_1 = \{-1,0,1\}^2 \setminus \big\{(0,0)\big\}$ is a root system:



△

Example. The following is a root system in $\mathbb{R}^4$:

$$\Phi_2 = \big\{e_i - e_j : 1 \leq i,j \leq 4, i \neq j\big\}$$

△

Example. The following is a root system in $\mathbb{R}^4$:

$$\Phi_3 = \big\{e_i - e_j, \pm e_i : 1 \leq i,j \leq 4, i \neq j\big\}$$

△

2.2 Simple Systems

A set $\Pi \subseteq \Phi$ is a *simple system* if

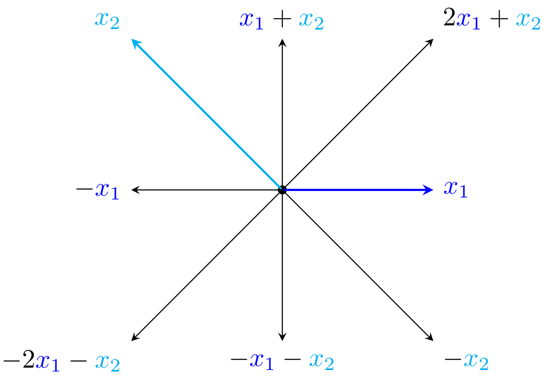
(*i*) Π is linearly independent;

(*ii*) For every $x \in \Phi$, $x = \sum_i \alpha_i y_i$ for some $y_i \in \Pi$ and $\alpha_i \in \mathbb{R}$ satisfying $\alpha_i \geq 0$ for all i, or $\alpha_i \leq 0$ for all i.

A simple system is similar to a basis in that it is linearly independent and it spans Φ, but with the stronger requirement that these linear combinations have all positive or all negative coefficients.

Example. A simple system for the root system Φ_1 defined in a previous example is given by

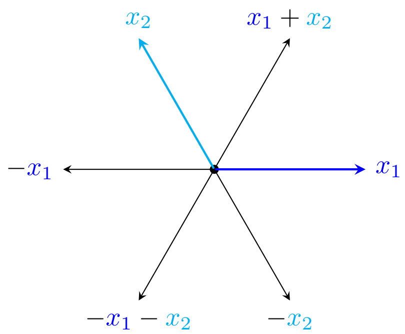
$$\Pi = \left\{x_1 = (1{,}0), x_2 = (-1{,}1)\right\}$$



△

Example. A simple system for $\Phi_{I_2(3)}$ is given by

$$\Pi = \left\{x_1 = (1{,}0), x_2 = (-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2})\right\}$$



with △

Example. A simple system for the root system Φ_2 defined in a previous example is given by

$$\Pi_2 = \{e_1 - e_2, e_2 - e_3, e_3 - e_4\}$$

△

Example. A simple system for the root system Φ_3 defined in a previous example is given by

$$\Pi_2 = \{e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4\}$$

△

In the examples above, we note that the angle between the roots in a simple system is obtuse (at least, for the examples we can visualise in $\mathbb{R}^2$). This turns out that this is a necessary condition for a set of roots to be a simple system:

Lemma 2.3. *Let Π be a simple system and suppose that $x,y \in \Pi$ are distinct. Then, $\langle x,y \rangle \leq 0$.*

Proof. Suppose otherwise that $\langle x,y \rangle > 0$. Then, $S_x(y) \in \Phi$ is given by $S_x(y) = y - 2\frac{\langle x,y \rangle}{\langle x,x \rangle}x = y - \alpha x$, where $\alpha = 2\frac{\langle x,y \rangle}{\langle x,x \rangle} < 0$.

Since Π is linearly independent, $1y - \alpha x$ is the unique representation of $S_x(y)$ as a linear combination of elements of Π. But then, we have coefficients $1 > 0$ and $-\alpha < 0$, contradicting that Π is simple. ∎

This lemma makes it slightly easier to construct simple systems: once the first vector has been chosen, we only have to consider vectors pointing in the opposite direction as candidates to be added to the simple system. For instance, in the previous example, once we have picked x_1, we only need to check the two vectors on the left (obviously excluding $-x_1$).

2.3 Ordered Vector Spaces

An *ordered vector space* is a vector space V equipped with a non-strict total ordering $\leq$ compatible with the vector space structure. That is:

- $\forall x,y,z \in V$, if $x \leq y$, then $x + z \leq y + z$ (compatibility with vector addition);
- $\forall x,y \in V, \forall \alpha \in \mathbb{R}$, if $x \leq y$ and $\alpha > 0$, then $\alpha x \leq \alpha y$;
- $\forall x,y \in V, \forall \alpha \in \mathbb{R}$, if $x \leq y$ and $\alpha < 0$ then $\alpha y \leq \alpha x$ (compatibility with scalar multiplication).

We write $x < y$ if $x \leq y$ and $x \neq y$.

The *lexicographical order* on $\mathbb{R}^n$ is given by $(x_1, \ldots, x_n) < (y_1, \ldots, y_n)$ if there exists $k \in \{1, \ldots, n\}$ such that $x_i = y_i$ for $i < k$ and $x_k < y_k$.

That is, compare the first components of the vectors using the usual ordering on $\mathbb{R}$; in case of a tie, compare the second components, and so on. In this way, the lexicographical ordering is a generalisation of dictionary ordering to non-alphabetical symbols.

Example. The vectors in $V = \{-1,0,1\}^2$ are lexicographically ordered as:

$$(-1,-1) < (-1,0) < (-1,1) < (0,-1) < (0,0) < (0,1) < (1,-1) < (1,0) < (1,1)$$

If we label -1 as a, 0 as b, and 1 as c and concatenate the components of each vector together, we have:

$$aa < ab < ac < ba < bb < bc < ca < cb < cc$$

matching the ordinary dictionary ordering of strings. △

Theorem 2.4. *Every possible total ordering on $\mathbb{R}^n$ is a lexicographical ordering for some basis.*

If $\leq$ is a total ordering on V, then for each $x \in V \setminus \{\mathbf{0}\}$, either $x < \mathbf{0} < -x$ or $-x < \mathbf{0} < x$, so every ordered vector space V can be partitioned into three sets: namely, the elements strictly less than 0, $V_- := \{x \in V : x < \mathbf{0}\}$, the elements strictly greater than 0, $V_+ := \{x \in V : x > \mathbf{0}\}$, and the singleton containing the zero vector, $\{\mathbf{0}\}$.

A *positive system* in a root system Φ is a subset $\Phi_+ \subset \Phi$ satisfying $\Phi_+ = \Phi \cap V_+$, where V_+ is induced by some total ordering on V. Similarly, a *negative system* is a subset $\Phi_- \subset \Phi$ such that $\Phi_- = \Phi \cap V_-$ for some total ordering on V.

Example. In the previous example, we saw the lexicographical ordering on $V = \{-1,0,1\}^2$. A positive system for the root system $\Phi_1 = V \setminus \{\mathbf{0}\}$ is then given by the elements greater than $\mathbf{0} = (0,0)$:

$$\Phi_+ = \big\{(0,1),(1,-1),(1,0),(1,1)\big\}$$

△

Example. Another positive system is given by

$$\Phi_+ = \big\{(1,-1),(1,0),(1,1),(0,-1)\big\}$$

with the ordering inducing the positive system given by the lexicographic ordering with respect to the basis $\big\{(1,-1),(0,-1)\big\}$. △

2.4 Quasisimple Systems

A subset $\Omega \subseteq \Phi_+$ is a *quasisimple system* if

(*i*) For each $x \in \Phi_+$, there exists a collection of scalar coefficients $\alpha_i \geq 0$ such that $x = \sum_i \alpha_i y_i$ for $y_i \in \Omega$;

(*ii*) Ω is minimal with respect to property (*i*).

Compared to simple systems, it is relatively easy to construct a quasisimple system:

Example. Consider the positive system

$$\Phi_+ = \big\{z_1 = (0,1), z_2 = (1,-1), z_3 = (1,0), z_4 = (1,1)\big\}$$

from a previous example.

Clearly, the whole set $\{z_1,z_2,z_3,z_4\}$ satisfies property (*i*).

But, $z_4 = z_1 + z_3$, so z_4 may be replaced in any linear combination with $z_1 + z_3$, and all the coefficients are still positive, so $\{z_1,z_2,z_3\}$ still satisfies (*i*).

Now, we note that $z_3 = z_1 + z_2$, so again, we may remove z_3 to obtain $\{z_1,z_2\}$.

At this point, we cannot remove any more vectors, so this set is minimal, and $\Omega = \{z_1,z_2\}$ is a quasisimple system. △

Lemma 2.5. *Let Ω be a quasisimple system and suppose that $x,y \in \Omega$ are distinct. Then, $\langle x,y\rangle \leq 0$.*

Proof. Suppose $\langle x,y\rangle > 0$. We have $S_x(y) \in \Phi$ and $S_x(y) = y - 2\frac{\langle x,y\rangle}{\langle x,x\rangle}x = y - \alpha x$, with $\alpha > 0$.

Suppose $S_x(y) \in \Phi_+$, so $S_x(y) = y - \alpha x = \sum_{z\in\Omega} \alpha_z z$ for some scalars $\alpha_z \geq 0$. Then,

$$y - \alpha x = \alpha_y y + \sum_{z\in\Omega\setminus\{y\}} \alpha_z z$$

$$(1-\alpha_y)y = \alpha x + \sum_{z\in\Omega\setminus\{y\}} \alpha_z z$$

if $\alpha_y < 1$, then dividing through by $1 - \alpha_y$ gives:

$$y = \frac{1}{1-\alpha_y}\left(\alpha x + \sum_{z\in\Omega\setminus\{y\}} \alpha_z z\right)$$

$$y = \frac{\alpha}{1-\alpha_y} + \sum_{z\in\Omega\setminus\{y\}} \frac{\alpha_z}{1-\alpha_y} z$$

so we can express y as a linear combination with positive coefficients, so $\Omega \setminus \{y\}$ is a quasisimple system, contradicting the minimality of Ω. So, $\alpha_y \geq 1$. Then,

$$\begin{aligned} y - \alpha_y y &= \alpha x + \sum_{z \in \Omega \setminus \{y\}} \alpha_z z \\ 0 &= (\alpha_y - 1)y + \alpha x + \sum_{z \in \Omega \setminus \{y\}} \alpha_z z \end{aligned}$$

All the coefficients on the right are non-negative, and $\Omega \subseteq \Phi_+$, so the right side is in V_+. Since $\alpha > 0$, αx is non-zero, and hence the right side is non-zero, which is a contradiction.

Otherwise, $S_x(y) \in \Phi_-$, so $S_x(y) = y - \alpha x = \sum_{z \in \Omega} -\alpha_z z$ for some scalars $\alpha_z \geq 0$. Then,

$$\begin{aligned} y - \alpha x &= \sum_{z \in \Omega} -\alpha_z z \\ \alpha x - y &= \sum_{z \in \Omega} \alpha_z z \\ x - \frac{1}{\alpha} y &= \sum_{z \in \Omega} \frac{\alpha_z}{\alpha} z \end{aligned}$$

so the previous argument applies, with the roles of x and y reversed. ∎

Theorem 2.6. *Every quasisimple system is a simple system.*

Proof. Every root in Φ_+ can be written as a non-negative linear combination L of roots in Ω. But then, every root in Φ_- can be written as the non-negative linear combination $-L$. So Ω satisfies property (*ii*) of a simple system.

We are left to show that Ω is linearly independent. Suppose there is a linear combination

$$\sum_{z \in \Omega} \alpha_z z = 0$$

with non-negative coefficients α_z not all zero. Define the sets

$$A := \{z \in \Omega : \alpha_z > 0\} \qquad B := \{z \in \Omega : \alpha_z < 0\}$$

and define the non-negative scalars $\beta_z := -\alpha_z$. Sorting positive and negative coefficients, we have

$$\begin{aligned} \sum_{\substack{z \in \Omega \\ \alpha_z > 0}} \alpha_z z + \sum_{\substack{z \in \Omega \\ \alpha_z < 0}} \alpha_z z &= 0 \\ \sum_{\substack{z \in \Omega \\ \alpha_z > 0}} \alpha_z z &= \sum_{\substack{z \in \Omega \\ \alpha_z < 0}} -\alpha_z z \\ \sum_{z \in A} \alpha_z z &= \sum_{z \in B} \beta_z z \end{aligned}$$

Define the vector y to be equal to these sums, which is non-negative. Then,

$$\begin{aligned} 0 &\leq \|y\|^2 \\ &= \langle y, y \rangle \\ &= \left\langle \sum_{z \in A} \alpha_z z, \sum_{z \in B} \beta_z z \right\rangle \end{aligned}$$

$$= \sum_{s \in B} \sum_{t \in A} \alpha_s \beta_t \langle s,t \rangle$$

The scalars are non-negative, and by the previous lemma, the forms are all non-positive, so the whole sum is non-negative:

$$\leq 0$$

so the coefficients must be all zero, contradicting their construction. ■

Theorem 2.7. *There is a bijection between positive systems and simple systems. Specifically, every positive system contains a unique simple system, and every simple system is contained within a unique positive system.*

Proof. Given a simple system Π, we can extend it to a basis B of V. Then, take the lexicographic order on V with respect to B. By construction, $\Pi \subset V_+$, so $\Pi \subset \Phi_+$.

Now, let Φ_+ be a positive system. Consider the set of all subsets of Φ_+ satisfying property (i) of a quasisimple system. Note that Φ_+ itself satisfies this property, so this set is non-negative. Now, choose one which is minimal, giving a quasisimple system, which is a simple system.

Uniqueness omitted. ■

Theorem 2.8. *Let $\Phi \supseteq \Phi_+ \supseteq \Pi$ be a root system, a positive system, and a simple system, respectively. Then, for all $x \in \Pi$ and all $y \in \Phi_+$:*

- *if $x \neq y$, then $S_x(y) \in \Phi_+$;*
- *if $x = y$, then $S_x(y) = -x \in \Phi_-$.*

Proof. (ii) follows from roots being negated under their associated reflections.

Otherwise, assume $x \neq y$, and let $y = \sum_{z \in \Pi} \alpha_z z$. Since $y \in \Phi_+$, $\alpha_z \geq 0$ for all z, and also since y is non-zero, at least one coefficient α_{z_0} is non-zero. Thus,

$$\begin{aligned} S_x(y) &= y - \alpha x \\ &= -\alpha x + \sum_{z \in \Pi} \alpha_z z \\ &= (\alpha_x - \alpha)x + \sum_{z \in \Pi \setminus \{x\}} \alpha_z z \end{aligned}$$

If $\alpha_x - \alpha > 0$, then this is a non-negative decomposition of $S_x(y)$, so $S_x(y) \in \Phi_+$.

Otherwise, $\alpha_x - \alpha < 0$, so this is a decomposition of $S_x(y)$ into a linear combination with both positive coefficients $\{\alpha_z\}_{z \neq x}$, and a negative coefficient $\alpha_x - \alpha$.

But $S_x(y) \in \Phi$, and Π is simple, so there also exists a non-negative or non-positive decomposition. So $S_x(y)$ has two distinct decompositions into linear combinations of vectors in Π, contradicting the linear independence of Π. ■

Intuitively, one might think that applying a reflection would perhaps swap all the vectors in the positive and negative half-spaces V_+ and V_-, or something similar. But, this theorem tells us that only one of these roots ever changes from being positive to negative: a root system captures a lot of information about the set of reflections its roots generate.

Given a root system Φ, we define its associated group G as

$$G = \langle S_x \mid x \in \Phi \rangle \leq O(V)$$

where V is the vector space containing Φ. This group acts on the root system Φ, since every element of G is a composition of reflections, and root systems are closed under reflection.

We claim that this group is finite.

Lemma 2.9. *If Φ_+ is a positive system, then $S_x(\Phi_+)$ is a positive system for any $x \in \Pi$.*

Lemma 2.10. *Let Π and Π' be simple systems in Φ. Then, there exists $g \in G$ such that $g(\Pi) = \Pi'$.*

Proof. Let Φ_+ and Φ'_+ be the associated unique positive systems for Π and Π, respectively, and let Φ_- and Φ'_- be the corresponding negative systems. We induct on $k := |\Phi_+ \cap \Phi_-|$.

If $k = 0$, then $\Phi_+ = \Phi'_+$, so $\Pi = \Pi' = \mathrm{id}_G(\Pi)$, since each positive system contains a unique simple system.

Assume the result holds for some arbitrary fixed $k \geq 0$. Then, $k + 1 \geq 1$, so $\Phi_+ \neq \Phi'_+$. Now, pick some $x \in \Pi \cap \Phi'_-$. Such an x exists, or else $\Pi \subseteq \Phi \setminus \Phi'_- = \Phi'_+$, so $\Pi = \Pi'$, contradicting the inductive hypothesis.

Now, consider $S_x(\Phi_+)$. By the previous theorem, every root apart from x is invariant under this reflection, and x alone is negated, so

$$S_x(\Phi_+) = \big(\Phi_+ \setminus \{x\}\big) \cup \{-x\}$$

Then,

$$S_x(\Phi_+) \cap \Phi'_- = (\Phi_+ \cap \Phi'_-) \setminus \{x\}$$

has cardinality k. ■

Theorem 2.11.

(*i*) *$G = \langle S_x \mid x \in \Pi \rangle$;*

(*ii*) *For all $y \in \Phi$, there exists $x \in \Pi$ and $g \in G$ such that $y = g(x)$.*

The first part of the theorem states that we can reduce the generating set from the entire root system Φ to just a simple system $\Pi \subseteq \Phi$. The second point says that every vector y in a root system is contained within a simple system $g(\Pi)$ for some $g \in G$. So in a way, a simple system contains almost as much information as the entire root system.

Proof. Let $x \in \Phi$, so $x = \sum_{r \in \Pi} \alpha_r r$. The *height* of x with respect to Π is defined as

$$h(x) := \sum_{r \in \Pi} \alpha_r$$

Define $G_0 = \langle S_x \mid x \in \Pi \rangle \leq G$. Then, for some fixed arbitrary $y \in \Phi_+$, define Λ_y to be the intersection of the orbit of y under G_0 with Φ_+:

$$\begin{aligned} \Lambda_y &:= G_0 \cdot y \cap \Phi_+ \\ &= \{g_0(y) : g_0 \in G_0\} \cap \Phi_+ \end{aligned}$$

Pick $z \in \Lambda_y$ with minimal height. Since $z \in \Phi_+$, we have

$$z = \sum_{x \in \Pi} \alpha_x x$$

with $\alpha_x \geq 0$ for all x. Then,

$$\begin{aligned} 0 &\leq \|z\|^2 \\ &= \langle z, z \rangle \end{aligned}$$

$$= \left\langle z, \sum_{x \in \Pi} \alpha_x x \right\rangle$$
$$= \sum_{x \in \Pi} \alpha_x \langle z,x \rangle$$

so $\langle z,x \rangle \geq 0$ for some $x \in \Pi$. Then,

$$\begin{aligned} S_x(z) &= z - 2\frac{\langle x,z \rangle}{\langle x,x \rangle} x \\ S_x(z) &= z - \alpha x \\ h\big(S_x(z)\big) &= h\big(z - \alpha x\big) \\ h\big(S_x(z)\big) &= h\big(z\big) - \alpha \end{aligned}$$

since z has minimal height in Λ_y, $S_x(z) \notin \Lambda_y$. Also, z is in the orbit $G_0 \cdot y$, so also $S_x(z) \in G_0 \cdot y$, and hence $S_x(z) \in \Phi_-$. But, the only root that can change sign under the reflection S_x is x, so $x = z$, and x is also in the orbit $G_0 \cdot y$. That is, there exists $g \in G_0$ such that $x = g \cdot y$, or $y = g^{-1} \cdot x$, proving *(ii)*.

Now, given $y \in \Phi$, let $x \in \Pi$ and $g \in G_0$ be such that $y = g \cdot x$. Then, $gS_xg^{-1} = S_{g(x)} = S_y$, so any reflection with a root in Φ can be expressed as the composition of a reflection S_x in $\Pi \subseteq G_0$ and two reflections $g,g^{-1} \in G_0$. So $G = G_0$. ■

The *length* of an element $g \in G$ is defined as

$$\ell(g) := \min\{n : \exists x_1,x_2,\ldots,x_n \in \Pi : g = S_{x_1}S_{x_2}\cdots S_{x_n}\}$$

That is, the length of an element g is the minimum number of reflections required to compose into g.

- $\ell(g) = 0$ if and only if $g = 1_G$;
- $\ell(S_x) = 1$ for all $x \in \Pi$.

While defined algebraically, this notion of length has geometric meaning, relating to root systems:

Theorem 2.12. *For all $g \in G$,*

$$\begin{aligned} \ell(g) &= \big|(g \cdot \Phi_+) \cap \Phi_-\big| \\ &= \big|\{x \in \Phi_+ : g \cdot x \in \Phi_-\}\big| \end{aligned}$$

That is, the length of g is equal to the number of positive roots that become negative when g is applied to them.

Proof. Define $N(g) := (g \cdot \Phi_+) \cap \Phi_-$, and $n(g) := |N(g)|$. The goal is to show that $n(g) = \ell(g)$.

We have:

- $n(g) = n(g^{-1})$, as $x \mapsto -g \cdot x$ is a bijection $N(g)$ to $N(g^{-1})$.
- For each $x \in \Pi$,
$$n(S_xg) = \begin{cases} n(g) + 1 & g^{-1} \cdot x \in \Phi_+ \\ n(g) - 1 & g^{-1} \cdot x \in \Phi_- \end{cases}$$
  and
$$n(S_xg^{-1}) = n(gS_x) = \begin{cases} n(g) + 1 & g \cdot x \in \Phi_+ \\ n(g) - 1 & g \cdot x \in \Phi_- \end{cases}$$

We show that $n(g) \leq \ell(g)$. Let $g = S_{x_1} S_{x_2} \cdots S_{x_k}$ with $k = \ell(g)$. Then,

$$\begin{aligned} n(g) &\leq n(S_{x_1} S_{x_2} \cdots S_{x_k}) \\ &\leq n(S_{x_1} S_{x_2} \cdots S_{x_{k-1}}) + 1 \\ &\leq n(S_{x_1} S_{x_2} \cdots S_{x_{k-2}}) + 2 \\ &\vdots \\ &\leq n(S_{x_1}) + (k-1) \\ &\leq k \\ &= \ell(g) \end{aligned}$$

Now, suppose $n(g) < k$, so there exists an index $i \leq k$ such that $n(S_{x_1} S_{x_2} \cdots S_{x_i}) = n(S_{x_1} S_{x_2} \cdots S_{x_{i-1}}) - 1$, so $S_{x_1} S_{x_2} \cdots S_{x_{i-1}}(x_i) \in \Phi_-$. Pick j to be the maximum index such that $S_{x_j} \cdots S_{x_{i-1}}(x_i) \in \Phi_-$, but $S_{x_{j+1}} \cdots S_{x_{i-1}}(x_i) \in \Phi_+$. Thus,

$$S_{x_{j+1}} \cdots S_{x_{i-1}}(x_i) = x_j$$

Let $h = S_{x_{j+1}} \cdots S_{x_{i-1}}$. Then, $h S_{x_i} h^{-1} = S_{x_j}$, so $S_{x_j} h S_{x_i} = h S_{x_i} h^{-1} h S_{x_i} = h$, so we can remove S_{x_j} and S_{x_i} from the decomposition of g, contradicting that $\ell(g) = k$. ■

Theorem 2.13. *Let Π and Π' be simple systems in Φ. Then, there exists a* ***unique*** *$g \in G$ such that $g\Pi = \Pi'$.*

Proof. Existence was proved in a previous theorem. For uniqueness, suppose $g \cdot \Pi = h \cdot \Pi = \Pi'$. Then, $h^{-1}g \cdot \Pi = \Pi$. Now, consider the positive root system Φ_+ associated to Π. Then, $\Phi_+ = h^{-1}g \cdot \Phi_+$, and hence $\ell(h^{-1}g) = 0$, so $h^{-1}g = 1_G$, and $g = h$.

■

Corollary 2.13.1. *There is a bijection between G and the set of simple systems in G.*

Proof. Orbit-stabiliser theorem. ■

Corollary 2.13.2. *Given a finite root system $\Phi \subseteq V$, $\big(\langle S_x \mid x \in \Phi \rangle, V\big)$ is a finite reflection group with order at most $2^{|\Phi|}$.*

So, we can convert a reflection group (G,V) into a root system $\Phi_{(G,V)}$, and this corollary tells us that we can recover (G,V) from $\Phi_{(G,V)}$.

Conversely, if we start with an abstract root system Φ, we can convert this into a reflection group (G,V), and from there, obtain the root system $\Phi_{(G,V)}$, which will be equal to the set of roots in Φ, each normalised to unit length.

3 Presentations of Groups

3.1 Free Groups

Given a set X, a *word* $w = x_1 x_2 \cdots x_n$ on X is a finite sequence of *letters* $(x_i)_{i=1}^n \subseteq X \cup X^{-1}$, where a letter is either an element of X or the formal inverse of an element of X, and we say that n is the *length* of the word. Note that the empty sequence of length 0 is a word, denoted by $\emptyset$.

The *concatenation* of two words $x_1 x_2 \cdots x_m$ and $y_1 y_2 \cdots y_n$ is the word $x_1 \cdots x_m y_1 \ldots y_n$.

A word w' is an *elementary contraction* of a word w if $w = y_1 x x^{-1} y_2$ and $w' = y_1 y_2$, where y_1, y_2 are (possibly empty) words and $x \in X \cup X^{-1}$, and we write $w \searrow w'$. We also say that w is an *elementary expansion* of w' and write $w' \nearrow w$.

Two words a and b are *equivalent* if there are words $w_1,\ldots,w_n$ such that $a = w_1$ and $b = w_n$ and for each i, either $w_i \nearrow w_{i+1}$ or $w_i \searrow w_{i+1}$.

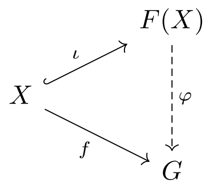
The *free group* $F(X)$ on the set X is the set of equivalence classes of words in X. The group operation is given by $[w] \cdot [w'] = [ww']$, and the identity element is given by $[\emptyset]$, also denoted by ε or e. The inverse of the element $[x_1x_2\cdots x_n]$ is given by $[x_m^{-1}\cdots x_2^{-1}x_1^{-1}]$.

A word is *reduced* if it does not admit an elementary contraction.

Theorem 3.1. *Every element of $F(X)$ is represented by a unique reduced word.*

3.2 Presentations

The free group satisfies a universal property in the category of groups, namely, given any function $f : X \to G$ from a set X to a group G, there is a unique homomorphism $\varphi : F(X) \to G$ $\varphi\big([x]\big) = f(x)$. That is, such that



commutes. That is, homomorphisms $F(X) \to G$ uniquely correspond to functions $X \to G$.

Because of this, given a group G, we can always find a set X and a surjection $F(X) \twoheadrightarrow G$.

Let G be a group and $B \subseteq G$ be a subset. The *normal subgroup generated by* B, denoted $\langle\langle B\rangle\rangle$, is smallest normal subgroup of G containing b. Or equivalently,

$$\langle\langle B\rangle\rangle \coloneqq \bigcap_{B\subseteq N\trianglelefteq G} N$$

Because the intersection of normal subgroups is a normal subgroup, $\langle\langle B\rangle\rangle$ is itself normal in G.

Lemma 3.2.

$$\langle\langle B\rangle\rangle = \left\{\prod_{i=1}^{n} g_i b^{\pm1} g_i^{-1} : n \in \mathbb{N}, b_i \in B, g_i \in G\right\}$$

If N is a normal subgroup of G containing B, then it certainly contains all the conjugates $g_i b_i^{\pm1} g_i^{-1}$, so N is a subset of this set. Conversely, this set contains B, as when $n = 0$ and $g_i = 1_G$, the product is just $b_i \in B$, and it can also be verified that this set is normal in G.

Let X be a set and $R \subseteq F(X)$. The group with *presentation* $\langle X \mid R\rangle$ is defined as

$$\langle X \mid R\rangle \coloneqq F(X)/\langle\langle R\rangle\rangle$$

Elements of the set R are called *relations*. Intuitively, the presentation $\langle X \mid R\rangle$ is the group on X that is as free as possible, subject to the constraint that every relation in R is identified with the identity.

Example.

- $\langle X \mid \emptyset\rangle \cong F(X)$
- $\langle t \mid t^n\rangle \cong \mathbb{Z}/n\mathbb{Z}$
- $\langle x,y \mid xyx^{-1}y^{-1}\rangle \cong \mathbb{Z}^2$

△

Example. The dihedral group of order $2n$ has presentation

$$\langle \sigma,\tau \mid \sigma^n,\tau^2,\tau\sigma\tau^{-1}\sigma \rangle$$

△

Because the relations are just specifying which elements are identified with the identity, we sometimes write equalities on the right side of a presentation to identify two expressions in the presentation. For instance, the more common presentation of the dihedral group of order $2n$ is given by

$$\langle \sigma,\tau \mid \sigma^n,\tau^2,\tau\sigma\tau^{-1} = \sigma^{-1} \rangle$$

Here, $\tau\sigma\tau^{-1} = \sigma^{-1}$ is called a *relator*, as it is not an element of R.

Example. The group with presentation

$$\langle a,b \mid ba^2b^{-1} = a^3,ab^2a^{-1} = b^3 \rangle$$

is the trivial group. △

As seen by this example, it is not immediately obvious what group any given presentation represents.

In fact, the *word problem* for a finitely generated group is the decision problem of determining whether two words in generators represent the same element. It turns out that the word problem is undecidable, so there is no algorithm to determine whether any given word is non-trivial.

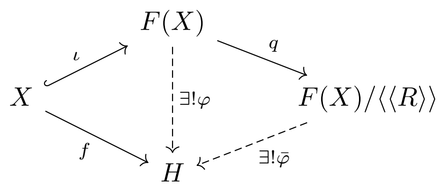
Lemma 3.3. *Let $G = \langle X \mid R \rangle$. If w and w' are two words in X, then $[w] = [w']$ if and only if one can be obtained from the other by a finite sequence of applications of:*

- *elementary contractions/expansions;*
- *inserting any relation $r \in R$, or its inverse, into one of the words.*

Obviously, adding or removing gg^{-1} or $g^{-1}g$ into a word does not change its equivalence class, as it still represents the same reduced word. Similarly, elements of G are words in X modulo relations, so adding relations into a word also does not change its equivalence class.

The useful property of group presentations is that it is easy to determine when a map is a homomorphism by using the universal property of the free group.

Lemma 3.4. *Let $G = \langle X \mid R \rangle$ and H be groups. Let $f : X \to H$ be a set function, and let φ be its unique extension from the universal property of the free group. Then, f descends to a homomorphism $\bar{\varphi} : G \to H$ if and only if $\varphi(r) = 1_H$ for all $r \in R$.*



Proof. If $\bar{\varphi}$ is a homomorphism, then it must send the identity to the identity, so every every element of $\langle\langle R \rangle\rangle \supseteq R$ must be sent to the identity.

Conversely, suppose $\varphi(r) = 1_H$ for all $r \in R$. Any element $s \in \langle\langle R \rangle\rangle$ can be expressed as

$$s = \prod_{i=1}^{n} s_i r_i^{\pm 1} s_i^{-1}$$

$$\varphi(s) = \varphi\left(\prod_{i=1}^{n} s_i r_i^{\pm 1} s_i^{-1}\right)$$

$$\varphi(s) = \prod_{i=1}^{n} \varphi(s_i)\varphi(r_i^{\pm 1})\varphi(s_i^{-1})$$
$$\varphi(s) = \prod_{i=1}^{n} \varphi(s_i)1_H^{\pm 1}\varphi(s_i^{-1})$$
$$\varphi(s) = \prod_{i=1}^{n} \varphi(s_i)\varphi(s_i)^{-1}$$
$$\varphi(s) = 1_H$$

so $s \in \ker(\varphi)$, and hence $\langle\langle R\rangle\rangle \subseteq \ker(\varphi)$, so this is a well-defined homomorphism from G to H. ■

Example. Consider the two presentations

$$G_1 = \langle x,y \mid xyx^{-1}y^{-1}\rangle, \qquad G_2 = \langle \sigma,\tau \mid \sigma^{2n},\tau^2,\tau\sigma\tau^{-1} = \sigma^{-1}\rangle$$

and define the set function $f : G_1 \to G_2$ on generators by

$$f(x) = \tau$$
$$f(y) = \sigma^n$$

Noormally, to verify that this defines a homomorphism, we would need to check that $f(\hat{x}\hat{y}) = f(\hat{x})f(\hat{y})$ for all words $\hat{x},\hat{y} \in G_1$. Because G_1 is an infinite group, this is difficult to do. However, the previous lemma tells us that we only need to verify that the relations are in the kernel:

$$\begin{aligned}
f(xyx^{-1}y^{-1}) &= \tau\sigma^n\tau^{-1}\sigma^{-n} \\
&= \tau\big[\sigma\cdots\sigma\big]\tau^{-1}\sigma^{-n} \\
&= \tau\big[\sigma(\tau^{-1}\tau)\sigma(\tau^{-1}\tau)\cdots(\tau^{-1}\tau)\sigma(\tau\tau^{-1})\sigma\big]\tau^{-1}\sigma^{-n} \\
&= (\tau\sigma\tau^{-1})(\tau\sigma\tau^{-1})\cdots(\tau\sigma\tau^{-1})(\tau\sigma\tau^{-1})\sigma^{-n} \\
&= (\tau\sigma\tau^{-1})^n\sigma^{-n} \\
&= a^{-n}a^{-n} \\
&= a^{-2n} \\
&= 1_H
\end{aligned}$$

△

4 Coxeter Groups

Coxeter groups are defined via either graphs or matrices, as defined here.

A *Coxeter matrix* $M = (m_{ij})$ is on a set X is an $|X| \times |X|$ matrix satisfying:

- $m_{ij} \in \mathbb{N}_{\geq 1} \cup \{\infty\}$;
- $m_{ij} = m_{ji}$;
- $m_{ij} = 1$ if and only if $i = j$.

That is, it is a symmetric matrix with 1 along the diagonal, and integers at least 2 or infinity in all other entries.

A *Coxeter graph* Γ is an undirected finite simple graph with edges labelled by elements of $\mathbb{N}_{\geq 3} \cup \{\infty\}$.

Theorem 4.1. *There is a one-to-one correspondence between Coxeter graphs and Coxeter matrices.*

Proof. Given a Coxeter matrix M on a set X, we construct the Coxeter graph on $n = |X|$ vertices labelled $\{1, \ldots, n\}$, where two vertices i,j are adjacent if and only if $m_{ij} \geq 3$.

Conversely, given a Coxeter graph G with vertex set V, we construct the $|V| \times |V|$ Coxeter matrix M by

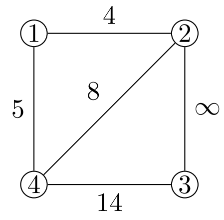
$$m_{ij} = \begin{cases} 1 & i = j \\ w(i,j) & \text{if edge } (i,j) \text{ exists} \\ 2 & \text{else} \end{cases}$$

■

Example. Given the Coxeter matrix

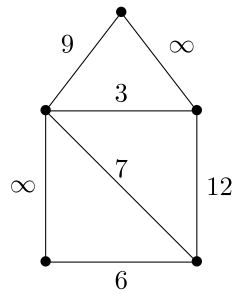
$$M = \begin{bmatrix} 1 & 4 & 2 & 5 \\ 4 & 1 & \infty & 8 \\ 2 & \infty & 1 & 14 \\ 5 & 8 & 14 & 1 \end{bmatrix}$$

the associated Coxeter graph is then

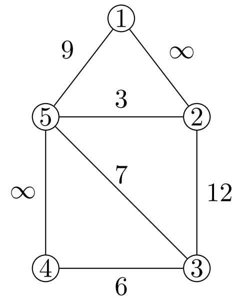


△

Example. Given the Coxeter graph



we first pick an ordering of the vertices:



then fill in the 5×5 matrix, with 1 on the diagonal; copying the edge weight of each edge; and filling in 2 otherwise:

$$M = \begin{bmatrix} 1 & \infty & 2 & 2 & 9 \\ \infty & 1 & 12 & 2 & 3 \\ 2 & 12 & 1 & 6 & 7 \\ 2 & 2 & 6 & 1 & \infty \\ 9 & 3 & 7 & \infty & 1 \end{bmatrix}$$

△

The matrix representation is more useful if the graph is very large, since lots of edges are hard to visualise. Conversely, if the graph is very sparse, and the matrix will be full of 2s and will be hard to read.

To simplify Coxeter graphs, it is convention to omit the label for edges with weight 3, since these edges will occur very frequently.

Given an $n \times n$ Coxeter matrix $M = (m_{ij})$ over a set X, the *Coxeter group* W_Γ is the group given by the presentation

$$W_\Gamma := \langle X \mid \forall i,j \leq n : (x_i x_j)^{m_{ij}} = 1 \rangle$$

- If $m_{ij} = \infty$, then there is no relation.
- If $i = j$, we have the relation $(x_i^2)^1 = x_i^2 = 1$ for every generator.
- If $m_{ij} = 2$, then $(x_i x_j)^2 = 1$, so

$$\begin{aligned} x_i x_j x_i x_j &= 1 \\ x_i x_j x_i x_j^2 &= x_j \\ x_i x_j x_i^2 &= x_j x_i \\ x_i x_j &= x_j x_i \end{aligned}$$

  so x_i and x_j commute.

Equivalently, given a Coxeter graph $\Gamma = (V,E)$ with associated Coxeter matrix $M = (m_{ij})$,the Coxeter group W_Γ defined by Γ is given by the presentation

$$W_\Gamma := \left\langle V \mid \forall i,j \in V : i^2,(ij)^{m_{ij}} \right\rangle$$

Lemma 4.2. *Let $G = \langle a_1, \dots ,a_n \rangle$ be a group generated by elements a_i all of order $|a_i| = 2$. Then, G is a quotient of the Coxeter group given by the Coxeter matrix with entries $m_{ij} = |a_i a_j|$.*

Proof. The Coxeter group given by this matrix has presentation

$$W = \left\langle x_1, \dots , x_n \mid (x_i x_j)^{|a_i a_j|} = 1 \right\rangle$$

Define the function $\varphi : W \to G$ on generators x_i by

$$\varphi(x_i) = a_i$$

We check that the relations are in the kernel of this map:

$$\begin{aligned} \varphi(x_i x_j) &= (a_i a_j)^{|a_i a_j|} \\ &= 1_G \end{aligned}$$

so φ defines a group homomorphism $W \to G$. ■

Theorem 4.3. *Let (G,V) be a finite reflection group with $G = \langle S_x \mid x \in \Pi \rangle$, and let W be the Coxeter group*

$$W = \left\langle x_1, \ldots, x_n \mid (x_i x_j)^{|S_i S_j|} = 1 \right\rangle$$

as defined in the previous proof. Then, the homomorphism $\varphi : W \to G$

$$\varphi(x_i) = S_i$$

as defined in the previous proof is an isomorphism.

So, not only can we reduce the generating set of a finite reflection group from an entire root system Φ to only a simple system Π, this theorem then says further that the only relations that are relevant are the orders of *pairs* of reflections. That is, there are no relations of the form $S_i S_j S_k \ldots = 1$.

Lemma 4.4 (Deletion Condition)**.** *Let (G,V) be a finite reflection group, and let $\Pi \subseteq \Phi_{(G,V)}$ be a simple system. Suppose $g = S_{x_1} S_{x_2} \cdots S_{x_n}$ for some roots $x_1, \ldots, x_n \in \Pi$, and $\ell(g) < n$. Then, there exist indices $1 \leq i < j \leq n$ such that*

$$g = S_{x_i} \cdots S_{x_{i-1}} S_{x_{i+1}} \cdots S_{x_{j-1}} S_{x_{j+1}} \cdots S_{x_n}$$

Example. There is a single Coxeter group on 1 generator, given by the presentation

$$\langle a \mid a^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$$

△

Example. For two generators a and b, there are two options, depending on the value of m_{ab}.

- If $m_{ab} = m < \infty$, then
$$\langle a,b \mid a^2, b^2, (ab)^m \rangle \cong \mathrm{Dih}(n)$$
is dihedral group of order $2n$.
- If $m_{ab} = \infty$, then
$$\langle a,b \mid a^2, b^2 \rangle \cong \mathrm{Dih}(\infty)$$
is the infinite dihedral group, which can be interpreted as the

△

Example. For three generators, a, b, and c, we have

$$W = \left\langle a,b,c \mid a^2, b^2, c^2, (ab)^k, (bc)^\ell, (ac)^m \right\rangle$$

If $k,\ell,m < \infty$, we have 3 cases:

- if $\frac{1}{k} + \frac{1}{\ell} + \frac{1}{m} = 1$, then this group describes the isometries of tilings of Euclidean 2-space, where each generator is a reflection;
- if $\frac{1}{k} + \frac{1}{\ell} + \frac{1}{m} < 1$, then this group describes the isometries of platonic solids, or the suspension of regular n-gons;
- if $\frac{1}{k} + \frac{1}{\ell} + \frac{1}{m} > 1$, then this group describes the isometries of tilings of hyperbolic 2-space.

where all the reflections meet at angles $\frac{\pi}{k}$, $\frac{\pi}{\ell}$, and $\frac{\pi}{m}$. △

4.1 Geometric Representations of Coxeter Groups

Given a Coxeter group W_Γ associated to a Coxeter graph $\Gamma = (V,E)$, the goal is to find a group homomorphism $\rho_\Gamma : W_\Gamma \to O\big(V_\Gamma,\langle\,\cdot\,,\cdot\,\rangle\big)$, where V_Γ is the $\mathbb{R}$-vector space with basis $\{e_i : i \in V\}$ given by vertex set V of Γ.

We define the symmetric bilinear form $\langle\,\cdot\,,\cdot\,\rangle_\Gamma$ on the basis vectors of V_Γ by:

$$\langle e_i,e_j\rangle = -\cos\left(\frac{\pi}{m_{i,j}}\right)$$

where $m_{ij} = w(i,j)$ is the weight of the edge (i,j) in Γ.

Note that

$$\langle e_i,e_i\rangle = -\cos\left(\frac{\pi}{1}\right) = 1$$

If $m_{ij} = 2$, then

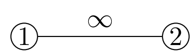
$$\langle e_i,e_j\rangle = -\cos\left(\frac{\pi}{2}\right) = 0$$

so e_i and e_j are orthogonal.

If $m_{ij} = \infty$, then

$$\langle e_i,e_j\rangle = -\cos\left(\frac{\pi}{\infty}\right) = -1$$

Consider the Coxeter graph



Then,

$$\begin{aligned}\langle e_1+e_2,e_1+e_2\rangle &= \langle e_1,e_1\rangle + 2\langle e_1,e_2\rangle + \langle e_2,e_2\rangle\\ &= 1-2+1\\ &= 0\end{aligned}$$

So, we have $\langle x,x\rangle = 0$ for $x \neq 0$, so this form is *not* positive definite, and hence $\big(V_\Gamma,\langle\,\cdot\,,\cdot\,\rangle\big)$ is not necessarily a Euclidean space.

One effect of this is that orthogonal complements do not behave as in Euclidean spaces. For instance, if $W \subseteq V$, then:

- $W^\perp \cap W = \{0_V\}$ does not hold;
- $W = (W^\perp)^\perp$ does not hold, but $W \subseteq (W^\perp)^\perp$ does;
- $V = W \oplus W^\perp$ does not hold.

Lemma 4.5. *Let $U \subseteq V$ be a finite-dimensional subspace. If $U \cap U^\perp = \{0_V\}$, then $V = U \oplus U^\perp$.*

Corollary 4.5.1. *If $x \in V$ satisfies $\langle x,x\rangle \neq 0$, then $V = \mathbb{R}x \oplus \{x\}^\perp$*

Theorem 4.6. *Let Γ be a Coxeter graph with two vertices x and y. Then, V_Γ is Euclidean if and only if $m_{xy} = w(x,y) < \infty$.*

Theorem 4.7. *Let $v = \alpha e_x + \beta e_y \in V_\Gamma$. Then,*

$$\begin{aligned}\langle v,v\rangle &= \alpha^2\langle e_x,e_x\rangle + 2\alpha\beta\langle e_x,e_y\rangle + \beta^2\langle e_y,e_y\rangle\\ &= \alpha^2 + 2\alpha\beta\cos\left(\frac{\pi}{m_{xy}}\right) + \beta^2\\ &\geq (\alpha-\beta)^2\end{aligned}$$

$$\geq 0$$

with equality if and only if $\alpha = \beta = 0$*, or if* $m_{xy} = \infty$ *and* $\alpha = \beta$*.*

We define a map $\rho_\Gamma : W_\Gamma \to O\big(V_\Gamma, \langle \cdot\,, \cdot \rangle\big)$ on generators by

$$\rho_\Gamma(x) = S_{e_x}$$

Lemma 4.8. *The composition* $S_{e_x} S_{e_y}$ *has order* m_{xy}*.*

Corollary 4.8.1. ρ_Γ *defines a group homomorphism.*

Corollary 4.8.2. *The element* xy *in* W_Γ *has order exactly* m_{xy}*.*

Proof. Since we have the relation $(xy)^{m_{xy}}$, the order of xy divides m_{xy}, and is therefore at most m_{xy}. However, its image under ρ_Γ has order m_{xy}, so it must have order at least m_{xy}. So $|xy| = m_{xy}$. ■

Recall that if a and b are vertices of Γ that are not connected by an edge, then $m_{ab} = 2$, so a and b commute in W_Γ.

Lemma 4.9. *Suppose* Γ *is disconnected, so* $\Gamma = \Gamma_1 \sqcup \Gamma_2$*. Then,*

$$\begin{aligned} W_\Gamma &\cong W_{\Gamma_1} \times W_{\Gamma_2} \\ V_\Gamma &\cong V_{\Gamma_1} \oplus V_{\Gamma_2} \\ \rho_\Gamma &\cong \rho_{\Gamma_1} \oplus \rho_{\Gamma_2} \end{aligned}$$

and V_{Γ_1} *is orthogonal to* V_{Γ_2}*.*

Lemma 4.10. *If* Γ *is a connected graph, then any* W_Γ*-invariant proper subspace is contained in* $V_\Gamma^\perp$*.*

Proof. Suppose $U \subseteq V_\Gamma$ is preserved by W_Γ. We claim that for each $x \in \Gamma$, we have either $e_x \in U$, or $U \subseteq e_x^\perp$.

Suppose $U \not\subseteq e_x^\perp$ so there exists $u \in U$ such that $\langle u, e_x \rangle \neq 0$. Then,

$$S_{e_x}(u) = u - 2\frac{\langle u, e_x \rangle}{\langle e_x, e_x \rangle} e_x$$

$$\frac{1}{2\langle u, e_x \rangle}\big(S_{e_x}(u) - u\big) = e_x$$

Since U is preserved by W_Γ, $S_{e_x}(u) \in U$, and also $u \in U$, so $e_x \in U$ as it is a linear combination of vectors in U.

This partitions the vertices of Γ into the sets

$$S_1 \coloneqq \{x \in \Gamma : e_x \in U\} \qquad S_2 \coloneqq \{x \in \Gamma : U \subseteq e_y^\perp\}$$

However, for all $x \in S_1$ and $y \in S_2$, we have $\langle e_x, e_y \rangle = 0$, so $m_{xy} = 2$, but Γ is connected. It follows that the vertices of Γ are contained entirely within one of the sets. If it is S_1, then $U = V_\Gamma$. Otherwise, if $U = S_2$ ■

Corollary 4.10.1. *If* Γ *is connected, and* V_Γ *is Euclidean, then* ρ_Γ *is irreducible. That is, there are no proper* W_Γ*-invariant subspaces.*

Let $\rho : G \to GL_n(\mathbb{R})$ be a group homomorphism. Then, ρ is *completely reducible* if there exist $\rho(G)$-invariant subspaces $V_1, \ldots, V_k$ such that $\mathbb{R}^n \cong V_1 \oplus \cdots \oplus V_k$, and G acting on V_i is irreducible.

In other words, there exists a basis of $\mathbb{R}^n$ such that the image of ρ is a block matrix with blocks along the diagonal and zero elsewhere.

Lemma 4.11. *If G is a finite group, then any representation $\rho : G \to GL_n(\mathbb{R})$ is completely reducible.*

Lemma 4.12. *If W_Γ is finite and Γ is connected, then ρ_Γ is irreducible.*

Suppose we have a group G acting on vector spaces X and Y. A linear map $f : X \to Y$ is *G-equivariant* if $f(g \cdot x) = g \cdot f(x)$. We denote the set of G-equivariant linear maps from X to Y by $\mathrm{Map}_G(X,Y)$. If $X = Y$, then these are G-equivariant endomorphisms and are denoted $\mathrm{End}_G(X)$.

Lemma 4.13. *Let Γ be connected and let W_Γ be finite. Then,* $\mathrm{End}_{W_\Gamma}(V_\Gamma) = \{k\,\mathrm{id}_{V_\Gamma} : k \in \mathbb{R}\} \cong \mathbb{R}$.

Theorem 4.14. *Suppose W_Γ is a finite group. Then $\big(V_\Gamma, \langle \cdot , \cdot \rangle_\Gamma\big)$ is Euclidean.*

Corollary 4.14.1. *If W_Γ is finite, then $\big(\rho_\Gamma(W_\Gamma), V_\Gamma\big)$ is a finite reflection group.*

So far, given a finite reflection group (G,V), we can find a Coxeter group W_Γ which is isomorphic to G.

Given a Coxeter group W_Γ, we have a representation that induces a finite reflection group $\big(\rho_\Gamma(W_\Gamma), V_\Gamma\big)$.

We will show that this representation ρ_Γ is faithful.

To do this we redefine some concepts for general Coxeter groups.

Let $g \in W_\Gamma$. Then, the *length* of g is

$$\ell(g) := \min\{n : \exists x_1, \ldots, x_n \in \Gamma : g = x_1 x_2 \cdots x_n\}$$

This satisfies similar properties to lengths for finite reflection groups:

- $\ell(g) = 0$ if and only if $g = 1_{W_\Gamma}$;
- $\ell(gh) \leq \ell(g) + \ell(h)$;
- $\ell(gh) \geq \ell(g) - \ell(h)$;
- for all $g \in W_\Gamma$ and $x \in \Gamma$, $\ell(gx) = \ell(g) + 1$ or $\ell(gx) = \ell(g) - 1$.

We abbreviate $\rho_\Gamma(x)$ to ρ_x.

Let W_Γ be a Coxeter group. The *root system* associated to W_Γ is defined by

$$\Phi_\Gamma := \big\{\rho_w(e_x) : w \in W_\Gamma, x \in \Gamma\big\}$$

That is, Φ_Γ is the union of the orbits of the basis vectors e_x.

A root is *positive* if it can be written as a non-negative linear combination of the e_x, and is *negative* if it can be written as a non-positive linear combination of the e_x.

Given a subset I of the vertices of Γ, the *parabolic subgroup* W_I of W_Γ corresponding to I is the subgroup of W_Γ generated by I. For $w \in W_I$, let $\ell_I(w)$ denote the length of w in the generating set I.

Theorem 4.15. *Let $g \in W_\Gamma$ and let x be a vertex of Γ. If $\ell(gx) > \ell(w)$, then $g \cdot e_x$ is a positive root. Similarly, if $\ell(gx) < \ell(w)$, then $g \cdot e_x$ is a negative root.*

Corollary 4.15.1. *Ever root in Φ is positive or negative.*

Theorem 4.16. *The representation ρ_Γ is faithful. That is,* $\ker(\rho_\Gamma) = \{1\}$.

Proof. If not, then let $g \in \ker(\rho_\Gamma)$ such that $\ell(w) > 1$. Then, there exists $x \in \Gamma$ such that $\ell(gx) < \ell(g)$. But then $e_x = g \cdot e_x$ must be a negative root ∎

Theorem 4.17. *The standard parabolic subgroup of W_Γ corresponding to I is isomorphic to the Coxeter group with vertex set I and labels coming from Γ.*

Corollary 4.17.1. *If W_Γ is finite, then $\Pi = \{e_x : x \in \Gamma\}$ is a simple system in Φ_Γ.*

A reflection group (G,V) is *essential* if V is the span of $\Phi_{(G,V)}$.

Theorem 4.18. *For any finite reflection group (G,V),*

$$(G,V) \simeq \big(G, \operatorname{span}(\Phi_{(G,V)}) \oplus U\big)$$

where G acts on U trivially.

Theorem 4.19. *The map $\Gamma \mapsto \big(\rho_\Gamma(W_\Gamma),V_\Gamma\big)$ is a bijection from the set of finite Coxeter graphs up to labelled graph isomorphisms, to the set of essential finite reflection groups.*

Note that

$$\Phi_{\left(\rho_\Gamma(W_\Gamma),V_\Gamma\right)} = \Phi_{W_\Gamma}$$

5 The Finiteness Criterion

The topology on $GL(V_\Gamma)$ comes from a norm on the set $\operatorname{End}(V_\Gamma)$ of all linear endomorphisms $T : V_\Gamma \to V_\Gamma$ as follows.

Let $\|\cdot\|$ be any norm on V_Γ, and for $T : V_\Gamma \to V_\Gamma$, define the operator norm by

$$\begin{aligned}\|T\| &:= \sup_{\|x\|=1} \|T(v)\| \\ &= \sup_{\|x\|\neq 0} \frac{\|T(v)\|}{\|v\|}\end{aligned}$$

The operator norm satisfies:

- $\|T(x)\| = \|T\|\|x\|$;
- $\|T\| = 0$ if and only if $T = 0$;
- $\|T + S\| \leq \|T\| + \|S\|$;
- if $T \in O(V_\Gamma)$, then $\|T\| = 1$;
- $O(V_\Gamma)$ is a closed, bounded, and compact subset of $\operatorname{End}(V_\Gamma)$.

Theorem 5.1. *Suppose that V_Γ is a Euclidean space. Then, W_Γ is a finite group.*

We have already proved the converse of this statement, so we have:

Corollary 5.1.1. *The Coxeter group W_Γ is finite if and only if V_Γ is Euclidean.*

A Coxeter graph Γ is *positive definite* if V_Γ is Euclidean. Everything we have done has allowed us to reduce to the case of understanding connected, positive definite Coxeter graphs.

A symmetric matrix is *positive definite* if the associated bilinear form defined by $\langle x,y\rangle = x^\top Ay$ is positive definite.

Lemma 5.2. *A symmetric matrix A is positive definite if and only if all of its eigenvalues are positive.*

Lemma 5.3. *Let $A = (a_{ij})_{1\leq i,j\leq n}$ be a symmetric matrix. Then, the associated form is positive definite if and only if for each $k \in \{1,\dots,n\}$, the upper left k-submatrix $A_k = (a_{ij})_{1\leq i,j\leq k}$ has positive determinant.*

Given a Coxeter graph $\Gamma = (V,E)$, and an induced Coxeter subgraph Λ with vertex set $I \subseteq V$, there is a natural inclusion $W_\Lambda \hookrightarrow W_\Gamma$, and the image of this map is the parabolic subgroup W_I, and we have $W_I \cong W_\Lambda$.

$$\begin{array}{ccc} W_\Lambda & \hookrightarrow & W_\Gamma \\ \downarrow{\scriptstyle \rho_\Lambda} & & \downarrow{\scriptstyle \rho_\Gamma} \\ GL(V_\Lambda) & \hookrightarrow & GL(V_\Gamma) \end{array}$$

Theorem 5.4. *If Λ is a Coxeter subgraph of Γ and:*

- *W_Γ is finite, then W_Λ is also finite;*
- *V_Γ is Euclidean, then V_Λ is also Euclidean;*
- *Γ is positive definite, then Λ is also positive definite.*

Let C_Γ be the matrix associated to the bilinear form on V_Γ. That is,

$$C_\Gamma = \left(-2\cos\left(\frac{\pi}{m_{xy}}\right)\right)_{x,y\in\Gamma}$$

We define $d(\Gamma) = \det(C_\Gamma)$.

Lemma 5.5. *Suppose Γ is a graph with a leaf node whose unique edge has label 3. Let Γ_1 be the graph obtained by deleting this node from Γ, and let Γ_2 be the graph obtained by deleting both endpoints of this edge. Then,*

$$d(\Gamma) = 2d(\Gamma_1) - d(\Gamma_2)$$

Proof. Order the vertices of the graph such that the last row in the Coxeter matrix for Γ corresponds to the leaf node, and the $(n-1)$ row corresponds to the other endpoint of the leaf node's edge:

$$M = \begin{bmatrix} & & & * & 0 \\ & C_{\Gamma_2} & & \vdots & \vdots \\ & & & * & 0 \\ * & \cdots & * & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$$

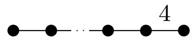
Laplacian expansion along the last row gives:

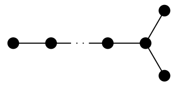
$$\begin{aligned} d(\Gamma) &= 2d(\Gamma_1) + (-1)^{(n+n-1)}(-1)\det \begin{bmatrix} & & & 0 \\ & C_{\Gamma_2} & & \vdots \\ & & & 0 \\ * & \cdots & * & -1 \end{bmatrix} \\ &= 2d(\Gamma_1) - d(\Gamma_2) \end{aligned}$$

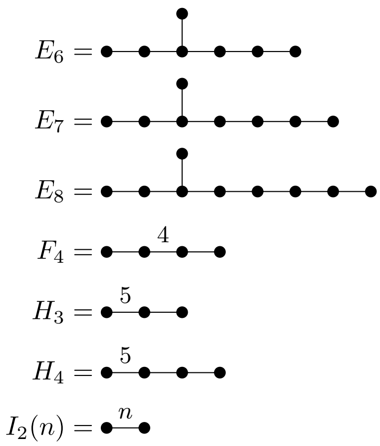
■

Theorem 5.6. *The following graphs have $d(\Gamma) > 0$:*

$A_n =$

$B_n =$ 

$D_n =$ 



where the subscript denotes the number of vertices in the Coxeter graph.

Theorem 5.7. *Suppose that Γ is a connected positive definite Coxeter graph. Then, Γ is one of the graphs above.*

6 The Exchange and Deletion Conditions

Let (W,S) be a pair consisting of a group W and a generating set S of elements of order 2. We say that (W,S) satisfies the *deletion condition* if, whenever $g = s_1 s_2 \cdots s_n$ for some$s_1, \ldots, s_n \in S$ with $\ell(g) < n$, there exists $1 \leq i < j \leq n$ such that

$$g = s_1 \cdots s_{i-1} s_{i+1} \cdots s_{j-1} s_{j+1} \cdots s_n$$

Note that the deletion condition depends on both the group and the generating set. For instance,

$$\big(\mathbb{Z}/2 \times \mathbb{Z}/2, \{(1,0),(0,1)\}\big)$$

satisfies the deletion condition, but

$$\big(\mathbb{Z}/2 \times \mathbb{Z}/2, \{(1,0),(0,1),(1,1)\}\big)$$

does not.

We have already seen that finite reflection groups satisfy this condition. To prove that this result also holds for Coxeter groups, we have to use the root system associated to the Coxeter group.

Recall that the root system for Γ is defined as $\Phi_\Gamma \coloneqq \{\rho_w(e_x) : w \in W_\Gamma, x \in \Gamma\}$. That is, the union of the orbits of the basis vectors e_x.

A root is *positive* if it can be written as a non-negative linear combination of the e_x, and is *negative* if it can be written as a non-positive linear combination of the e_x.

Lemma 6.1. *Let $x \in \Gamma$. Then, $\rho_x(\Phi_+) \cap \Phi_- = \{e_x\}$.*

Theorem 6.2. *Let $\Gamma = (V,E)$ be a Coxeter graph. Then, (W_Γ, V) satisfies the deletion condition.*

Let (W,S) be a pair consisting of a group W and a generating set S of elements of order 2. We say that (W,S) satisfies the *exchange condition* if: whenever $s_1 \cdots s_r = t_1 \cdots t_r$ are two words in S representing the same element $w \in W$ with $\ell(w) = r$ and $s_1 \neq t_1$, then there is an index $i \in \{2, \ldots, r\}$ such that $w = s_1 t_1 \cdots t_{i-1} t_{i+1} \cdots t_r$.

Theorem 6.3. *Suppose (W,S) satisfies the deletion condition. Then, (W,S) satisfies the exchange condition.*

Corollary 6.3.1. *Let W_Γ be a Coxeter group. Then, the pair (W_Γ, Γ) satisfies the exchange condition.*

Given a pair (W, S) satisfying the exchange condition, we can construct a Coxeter graph with vertex set S and edge weights $m_{st} = |st|$ to be the order of the word st. This yields a Coxeter group that surjects onto W. Let $M = (m_{ij})$ be the Coxeter matrix associated with this Coxeter group. Then, an *M-elementary reduction* of a word $w \in W$ is one of the following operations:

- Delete a subword of the form ss for $s \in S$;
- Replace a subword $sts\cdots$ with $tst\cdots$ where each of the words has exactly $m_{st} = |st|$ letters.

Theorem 6.4. *Let (W, S) be a pair satisfying the deletion condition, and let M be the associated Coxeter matrix. Let $w = s_1 \cdots s_k$ be a word with length $\ell(w) = k$. Then, given any other decomposition $w = t_1 \cdots t_m$, we can obtain $s_1 \cdots s_k$ from $t_1 \cdots t_m$ using M-elementary reductions.*

Corollary 6.4.1. *If (W, S) is a pair satisfying the deletion condition, then W is a Coxeter group.*

7 The Davis Complex

If W_Γ is finite, then (W_Γ, V_Γ) is a finite reflection group, so it has a nice group action on Euclidean space. Furthermore, this action preserves the unit sphere, and this restricts to a nice action on S^n.

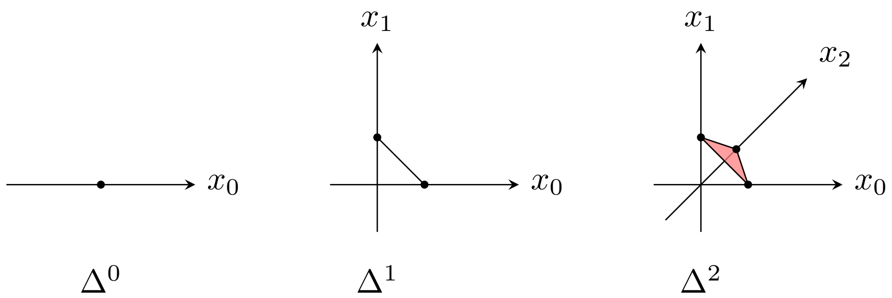
For infinite Coxeter groups, the geometric representation gives a faithful action on the inner product space $\big(V_\Gamma, \langle \cdot\,, \cdot \rangle_\Gamma\big)$. For the infinite case, there is a "nicer" space upon which the Coxeter group acts, and this is known as the *Davis complex.*

7.1 Simplicial Complexes

The *standard n-simplex* $\Delta^n \subseteq \mathbb{R}^{n+1}$ is the subspace

$$\Delta^n := \left\{ \mathbf{x} \in \mathbb{R}^{n+1} : x_i \geq 0, \sum_{i=0}^{n} x_i = 1 \right\}$$

whose vertices $v_0, v_1, \ldots, v_n$ are the unit vectors along the coordinate axes.



The standard n-simplex for $n = 0,1,2$

The *vertice set* $V(\Delta^n)$ of the n-simplex is the set of points where $x_i = 1$ for some i, and we denote the vertex corresponding to $x_i = 1$ by v_i. Each dimension adds an additional vertex, so $|V(\Delta^n)| = n + 1$. Also note that $V(\Delta^n)$ forms a basis for $\mathbb{R}^{n+1}$.

For each non-emptyset $A \subseteq \{0, \ldots, n\}$, we define a *face* Δ_A of Δ^n to be the subspace

$$\Delta_A := \big\{ (x_0, \ldots, x_n) \in \Delta^n : \forall i \notin A, x_i = 0 \big\}$$

Note that we consider Δ^n to be a face of itself.

We define the *interior* $\mathring{\Delta}^n$ of the n-simplex to be the subspace of points where $x_i > 0$ for all i. Note that for $n = 0$, we have $\mathring{\Delta}^0 = \Delta^0$.

Suppose $m \leq n$ and suppose we have an injection $f : \{0,\ldots,m\} \to \{0,\ldots,n\}$. Then, this map extends to a map $f_* : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ given by

$$f_*(v_i) = v_{f(i)}$$

for the bases $V(\Delta^m)$ and $V(\Delta^n)$. This induces a continuous map $\Delta^m \to \Delta^n$, which we call a *face inclusion*.

An *abstract simplicial complex* is a pair $K = (V,\Sigma)$, where V is a set containing the *vertices* of K, and Σ is a set of finite subsets of V containing the *simplices* of K, satisfying:

- for each $v \in V$, $\{v\} \in \Sigma$;
- if $\sigma \in \Sigma$ and $\tau \subseteq \sigma$, then $\tau \in \Sigma$ (transitive);

We can associate to each abstract simplicial complex a topological space.

The *topological realisation* $|K|$ of an abstract simplicial complex $K = (V,\Sigma)$ is obtained as follows:

1. For each $\sigma \in \Sigma$, take a copy Δ^n_σ of the standard n-simplex, where $n = |\sigma| - 1$, and pick a bijection $V(\Delta^n_\sigma) \to \sigma$.
2. Whenever $\tau \subset \sigma$, using the above bijections, obtain an injection $V(\Delta_\tau) \to V(\Delta_\sigma)$, inducing a face inclusion $f_{\tau\sigma} : \Delta_\tau \to \Delta_\sigma$.
3. Define
$$|K| = \left(\bigsqcup_{\sigma\in\Sigma} \Delta_\sigma \right) / \sim$$
where $x \sim f_{\tau\sigma}(x)$ for all $x \in \Delta_\tau$ and all $\tau \subset \sigma$.

That is, $|K|$ is the disjoint union of the simplices modulo the face inclusions.

Example. If $K_X = \big(X,\mathcal{P}(X)\big)$, then $|K_X| \cong \Delta^{|X|-1}$. △

For each $\sigma \in \Sigma$, there is a natural map $\Delta_\sigma \to |K|$ that identifies Δ_σ with a subspace of $|K|$. Given $\tau \neq \sigma$, we see that $\mathring{\Delta}_\sigma \cap \mathring{\Delta}_\tau = \emptyset$. Thus, for each point $x \in |K|$, there is a unique $\sigma \in \Sigma$ such that $x \in \mathring{\Delta}_\sigma$.

7.2 Geometric Realisations of Posets

Recall that a *partially ordered set* or *poset* is a set equipped with a reflexive, symmetric, and transitive relation, or a *partial ordering*, $\leq$.

Given a poset $(X, \leq)$, we can associate a simplicial complex $K_X = (X,\Sigma)$, where Σ consists of the subsets $\{x_0,\ldots,x_n\}$ such that $x_i \leq x_j$ if $i \leq j$. We call $|K_X|$ the *geometric realisation* of X.

Given a Coxeter graph Γ, we can consider the set $P_\Gamma := \{I \subseteq \Gamma : W_I \text{ is finite}\}$. This is a poset when ordered by inclusion. Note that $\emptyset \in P_\Gamma$ since $W_\emptyset = \{1\}$.

Denote the geometric realisation of P_Γ by K_Γ. As before, for each point $x \in |K|$, there is a unique $\sigma \in \Sigma$ such that $x \in \mathring{\Delta}_\sigma$. Since a simplex σ corresponds to a chain in P_Γ, there is a minimal element in σ, which we will denote by I_x. We define the *point stabiliser* of x to be W_{I_x}, also denoted by W_x when no confusion will arise.

The *Davis complex* Σ_Γ associated to Γ is the space $K_\Gamma \times W_\Gamma / \sim$, where $(x,w) \sim (x',w')$ if $x = x'$ and $w^{-1}w' \in W_x$.

Lemma 7.1. *The Davis complex Σ_Γ has an action of W_Γ given by $w \cdot [(x,z)] = [(x,wz)]$.*

Lemma 7.2. *The stabiliser of the point $[(x,z)]$ is given by zW_xz^{-1}.*